NANYANG TECHNOLOGICAL UNIVERSITY

Fluid mechanics and some novel surface waves
Part I

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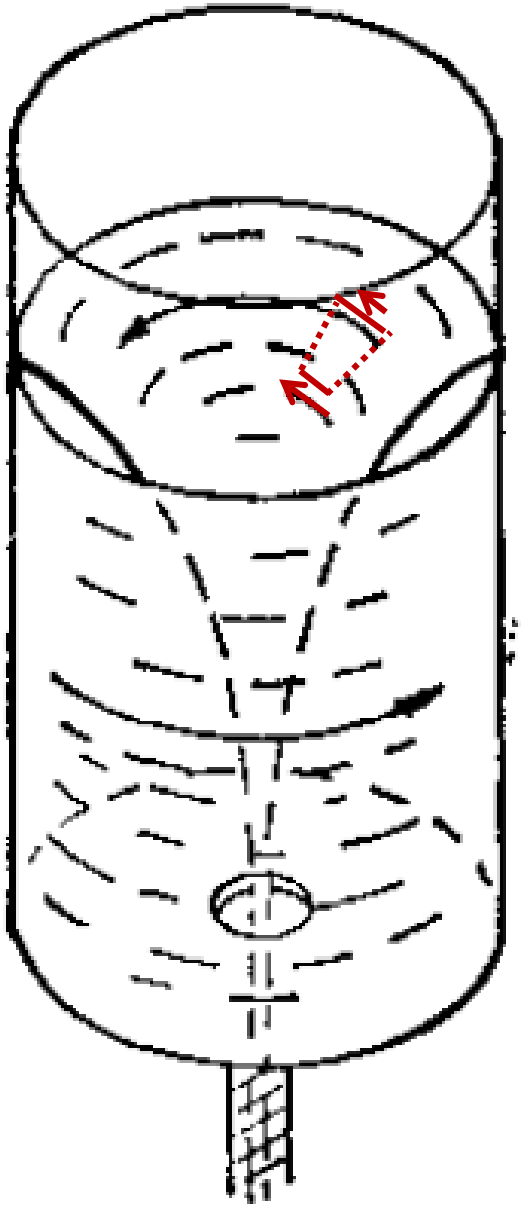
Bathtub drain vortex

Ahmad Falahatpisheh and
Arash Kheradvar, University of
California, Irvine

Bathtub Vortex



Feynman Lectures, Vol. 2

Curl-free bathtub vortex

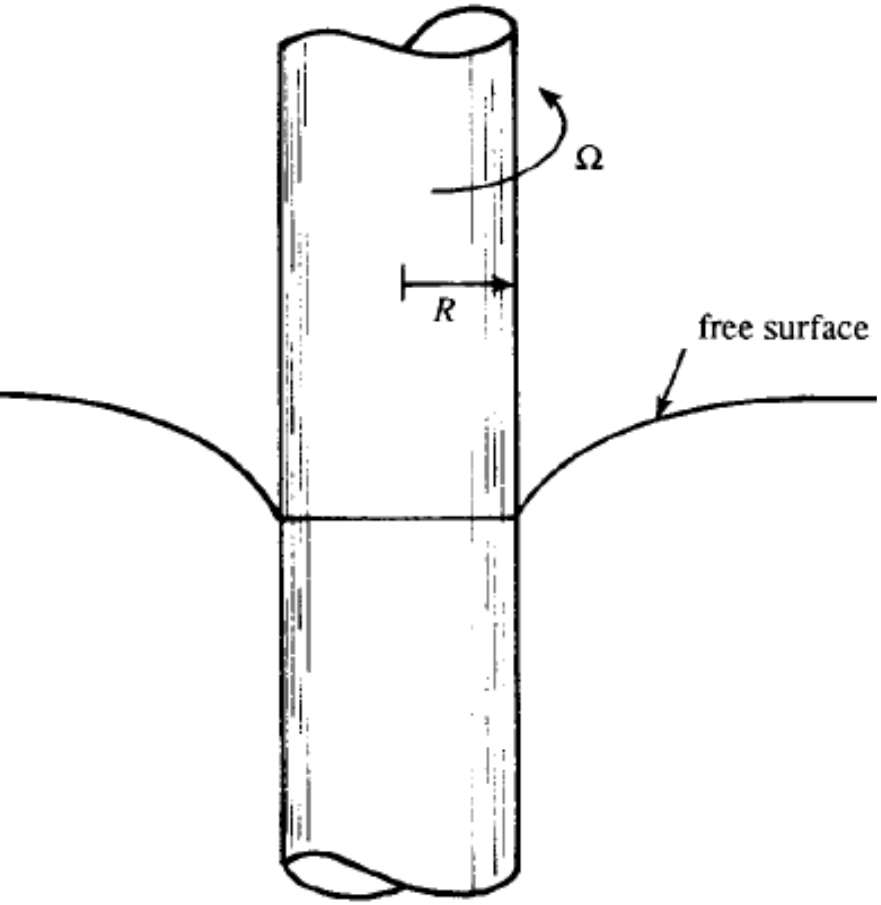
$$\nabla \times \vec{v} = 0$$

$$\Rightarrow \quad v = \frac{C}{2\pi r}$$

$$\Rightarrow \quad \oint \vec{v} \cdot ds = 2\pi r v = C$$

Circulation

Bathtub Vortex



Curl-free bathtub vortex

$$\nabla \times \vec{v} = 0$$

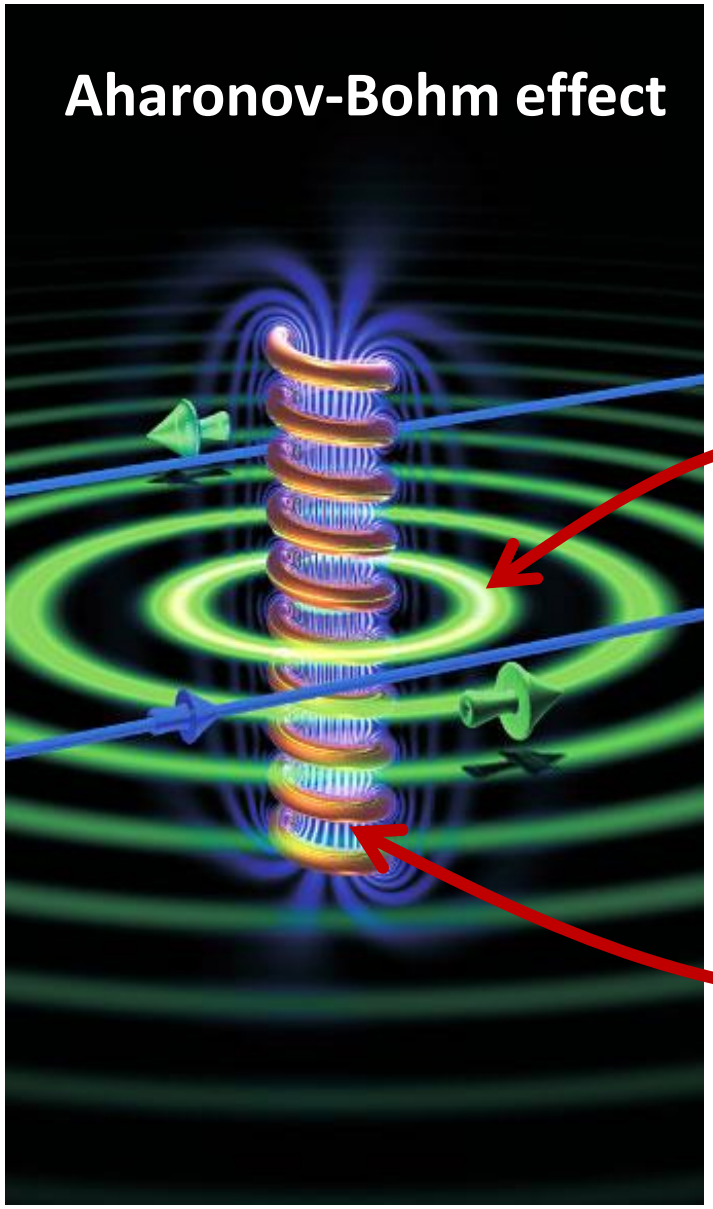
$$\Rightarrow \quad v = \frac{C}{2\pi r} \quad > R$$

$$\Rightarrow \quad \oint \vec{v} \cdot ds = 2\pi r v = C$$

Circulation

Kundu and Cohen, Fluid Mechanics, 3rd edition

Bathtub Vortex vs AB Effect



Curl-free bathtub vortex

$$\nabla \times \vec{v} = 0$$

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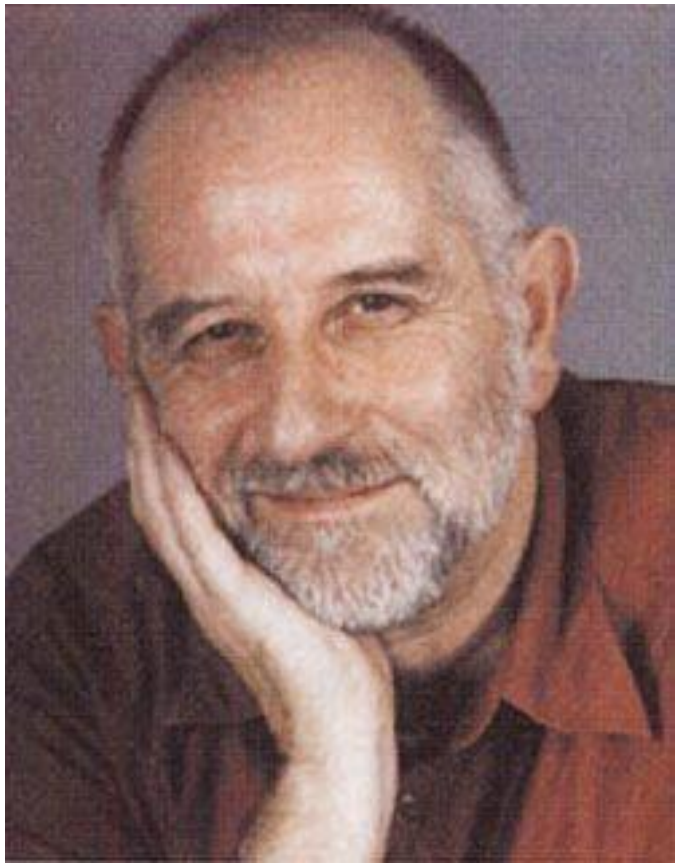
Circulation

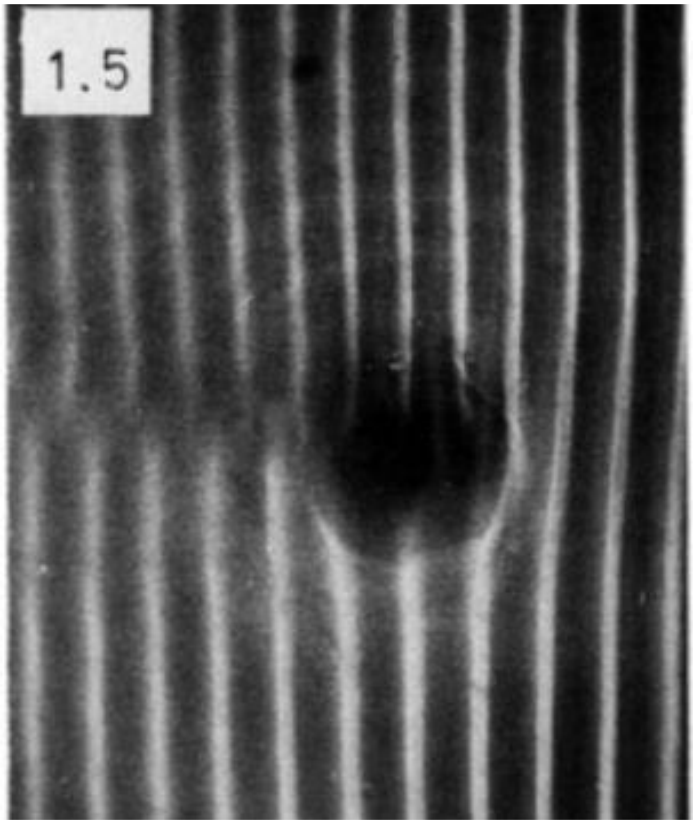
Physics Today 62, 38 (2009)/AIP

Bathtub Vortex vs AB Effect



M. V. Berry

Wavefront dislocations



Eur. J. Phys. 1, 154-162 (1980)

‘Vector Potential’ in k-Space

In the continuum

$$\vec{p} = -i\hbar\vec{\partial}_r \qquad \vec{r} = i\hbar\vec{\partial}_p$$

$$\vec{k} = -i\vec{\partial}_r \qquad \vec{r} = i\vec{\partial}_k$$

In a lattice

$$\vec{r} = i\vec{\partial}_k$$ + ?

For a charge in EM field

$$\vec{k} = -i\vec{\partial}_r - \frac{e}{c}\vec{A}$$

“Minimal coupling”

‘Vector Potential’ in k-Space

In the continuum $\vec{r} = i\vec{\partial}_k$

$$\Psi(\vec{r}) = \int d\vec{k}\Psi(\vec{k})e^{i\vec{k}\cdot\vec{r}}$$

$$\vec{r}\Psi(\vec{r}) = \int d\vec{k}\vec{r}\Psi(\vec{k})e^{i\vec{k}\cdot\vec{r}}$$

$$= \int d\vec{k}\Psi(\vec{k})[-i\vec{\partial}_k e^{i\vec{k}\cdot\vec{r}}]$$

$$= \int d\vec{k}i\vec{\partial}_k\Psi(\vec{k})e^{i\vec{k}\cdot\vec{r}}$$

$$\Rightarrow \quad \vec{r} = i\vec{\partial}_k$$

‘Vector Potential’ in k-Space

In a lattice $\vec{r} = i\vec{\partial}_k$ + ?

$$\Psi(\vec{r}) = \sum_n \int d\vec{k} \Psi_n(\vec{k}) \psi_{n,k}(\vec{r}) \qquad \psi_{n,k}(\vec{r}) = u_{n,k}(\vec{r}) e^{i\vec{k}\cdot\vec{r}}$$

$$\vec{r}\Psi(\vec{r}) = \sum_n \int d\vec{k} \Psi_n(\vec{k}) u_{n,k}(\vec{r}) (-i\vec{\partial}_k e^{i\vec{k}\cdot\vec{r}})$$

$$= \sum_n \int d\vec{k} [i\vec{\partial}_k \Psi_n(\vec{k}) u_{n,k}(\vec{r}) + \Psi_n(\vec{k}) i\vec{\partial}_k u_{n,k}(\vec{r})] e^{i\vec{k}\cdot\vec{r}}$$

$$= \sum_n \int d\vec{k} [i\vec{\partial}_k \Psi_n(\vec{k}) u_{n,k}(\vec{r}) + \Psi_n(\vec{k}) \int dr' \delta(r - r') i\vec{\partial}_k u_{n,k}(\vec{r}')] e^{i\vec{k}\cdot\vec{r}}$$

$$= \sum_n \int d\vec{k} [i\vec{\partial}_k \Psi_n(\vec{k}) u_{n,k}(\vec{r}) + \Psi_n(\vec{k}) \int dr' \sum_m u^*_{m,k}(r') u_{m,k}(r) i\vec{\partial}_k u_{n,k}(\vec{r}')] e^{i\vec{k}\cdot\vec{r}}$$

(Define $i\int dr' u^*_{m,k}(r')\vec{\partial}_k u_{n,k}(r') = -\mathcal{A}_{m,n}$)

$$= \sum_n \int d\vec{k} [i\vec{\partial}_k - \sum_m \mathcal{A}_{m,n}] \Psi_n(\vec{k}) \psi_{n,k}(\vec{r})$$

‘Vector Potential’ in k-Space

$$\vec{r}\Psi(\vec{r}) = \sum_n \int d\vec{k}[i\vec{\partial}_k - \sum_m \mathcal{A}_{m,n}]\Psi_n(\vec{k})\psi_{n,k}(\vec{r})$$

where $\mathcal{A}_{m,n} = -i\int dr' u^*_{m,k}(r')\vec{\partial}_k u_{n,k}(r')$

If only a single band is considered

$$\vec{r}\Psi(\vec{r}) = \int d\vec{k}[i\vec{\partial}_k - \mathcal{A}_{n,n}]\Psi_n(\vec{k})\psi_{n,k}(\vec{r})$$

So $\vec{r} = i\vec{\partial}_k - \mathcal{A}_n$

Berry connection

$$\mathcal{A}_n = -i < u_{n,k} | \nabla_k | u_{n,k} >$$

$$\vec{k} = -i\vec{\partial}_r - \frac{e}{c}\vec{A}$$

Vector potential



Potential' in k-Space

Berry connection

$$\vec{\mathcal{A}}_n$$

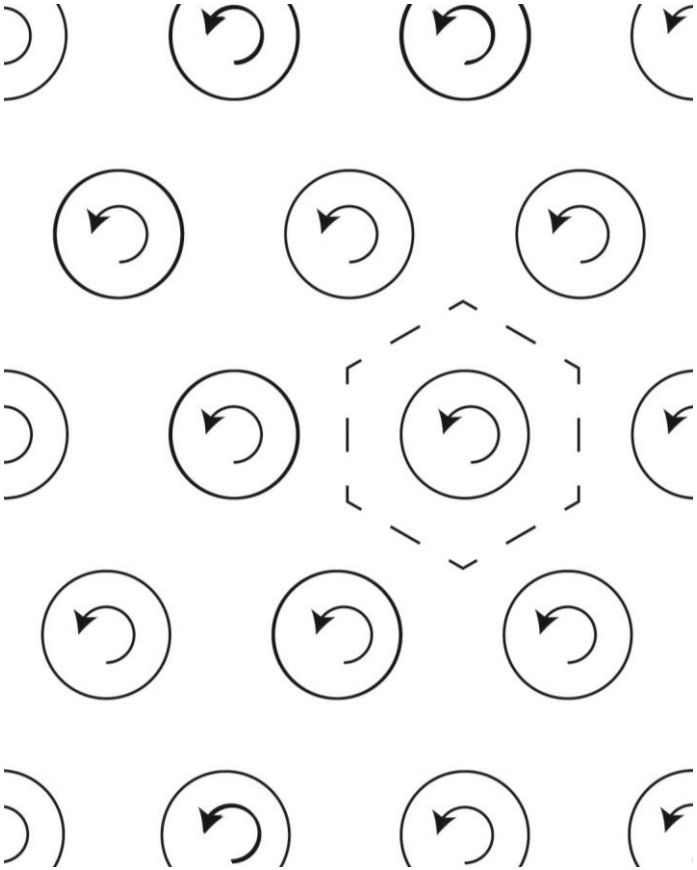
Berry curvature

$$\vec{\mathcal{F}}_n = \nabla_k \times \vec{\mathcal{A}}_n$$

Chern number

$$\oint\oint_{BZ} \vec{\mathcal{F}} \cdot d\vec{k} = 2\pi C$$

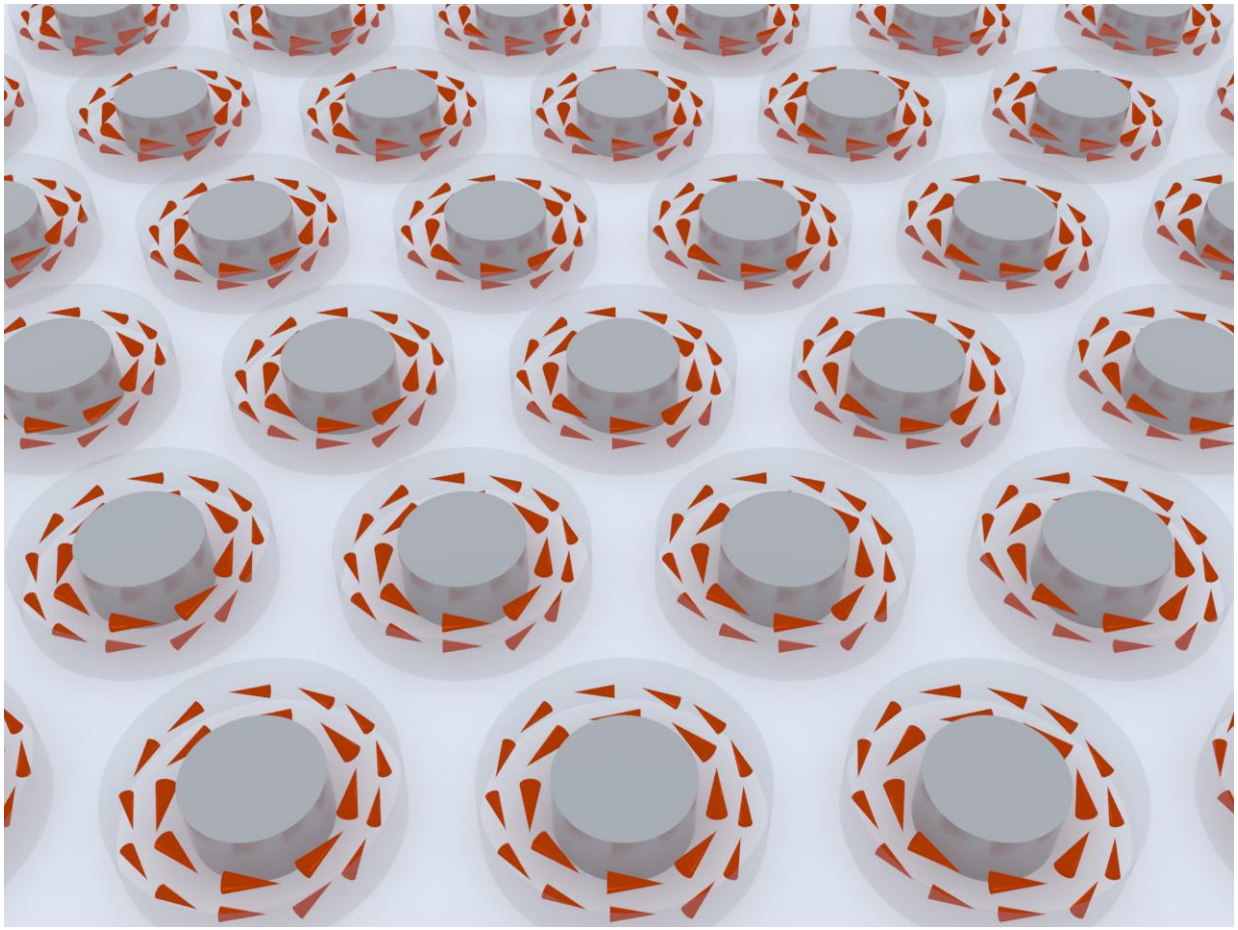
A Lattice of Vortices?

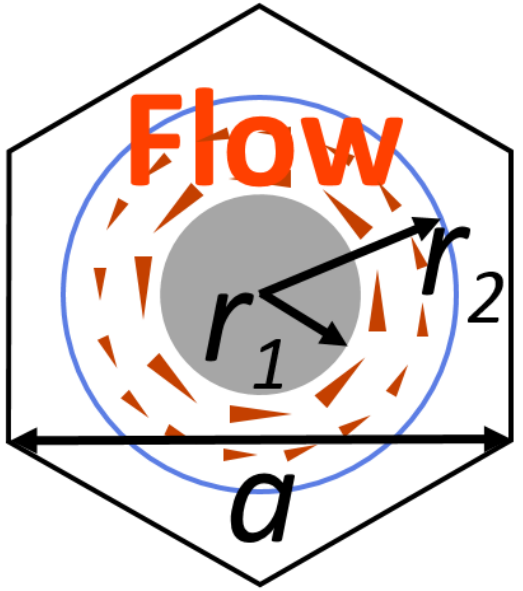


Difficulty: It's not a periodic system

Bloch theorem not applicable

A Lattice of Confined Vortices





At $r = r_1$

$$v_\theta = \Omega r_1$$

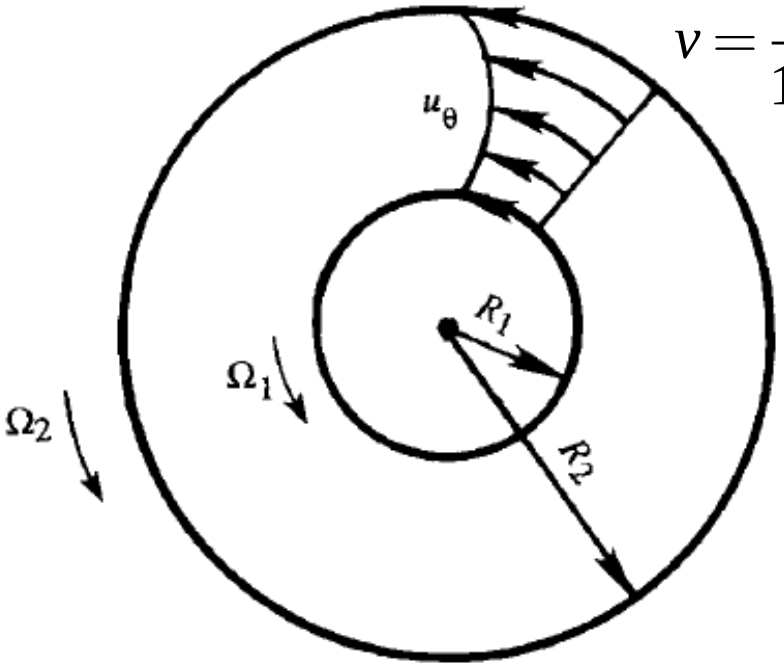
At $r = r_2$

$$v_\theta = 0$$

PRL 114, 114301 (2015)

Circular Couette Flow

Analytic solution from Navier-Stokes equations



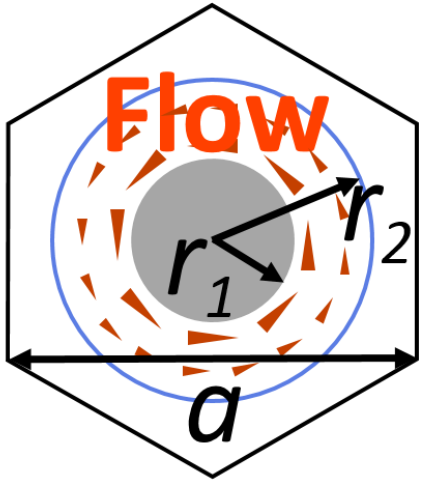
$$v = \frac{1}{1-(R_1/R_2)^2}\left\{\left[\Omega_2 - \Omega_1\left(\frac{R_1}{R_2}\right)^2\right]r + \frac{R_1^2}{r}\ \Omega_1 - \Omega_2\right\}$$

When $\Omega_2 = 0$

$$v = \frac{1}{1-(R_1/R_2)^2}\left\{-\Omega_1\left(\frac{R_1}{R_2}\right)^2 r + \frac{R_1^2}{r}\Omega_1\right\}$$

Kundu and Cohen, Fluid Mechanics, 3rd edition

Acoustic Waves with 'Magnetic Flux'



$$\tilde{v} = v_0 + v$$

$$\tilde{\rho} = \rho + \rho'$$

$$\tilde{p} = p_0 + p$$

$$p = \rho(\partial_t + \vec{v}_0 \cdot \nabla)\phi$$

Euler's equation:

$$\partial_t \tilde{v} + \tilde{v} \cdot \nabla \tilde{v} = -1/\tilde{\rho} \nabla \tilde{p}$$

Continuity:

$$\partial_t \tilde{\rho} + \nabla \cdot (\tilde{\rho} \tilde{v}) = 0$$

Master equation

$$\frac{1}{\rho} \nabla \cdot \rho \nabla \phi - (\partial_t + \vec{v}_0 \cdot \nabla) \frac{1}{c^2} (\partial_t + \vec{v}_0 \cdot \nabla)\phi = 0$$

J. Acoust. Soc. Am. 87 (6), 2292 (1990).

At $r = r_1$ $\Rightarrow$ $\vec{n} \cdot \nabla \phi = 0$

Let $\Psi = \phi \sqrt{\rho}$

Schrodinger-type

$\Rightarrow$ $[(\nabla - i\vec{A}_{eff})^2 + V(x,y)]\Psi = 0$

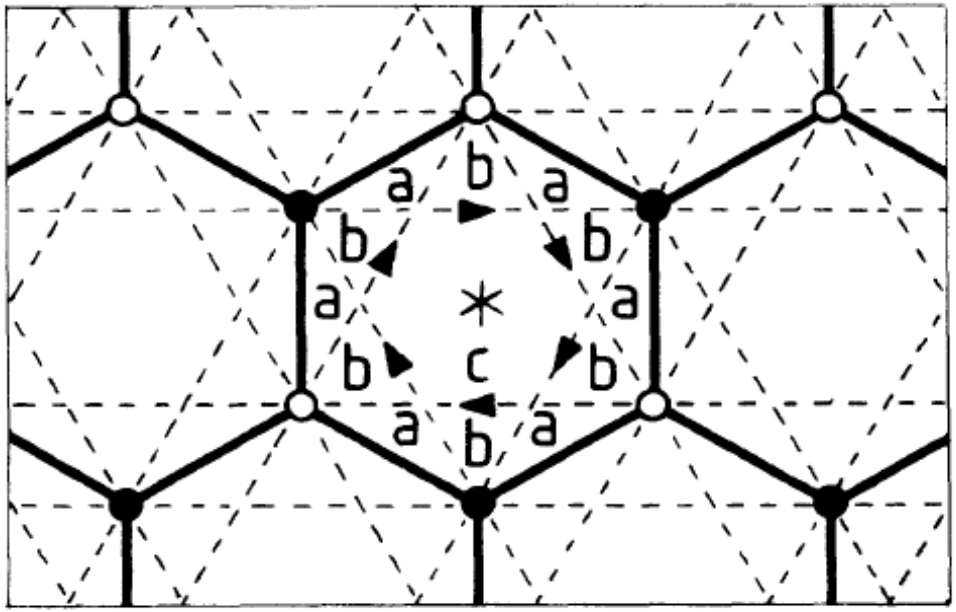
where

$$\vec{A}_{eff} = -\frac{\omega \vec{v}_0(x,y)}{c^2}$$

$$V(x,y) = -\frac{1}{4}\left|\nabla \ln \rho\right|^2 - \frac{1}{2}\nabla^2 \ln \rho + \frac{\omega^2}{c^2}$$

Haldane Model with Staggered Magnetic Field

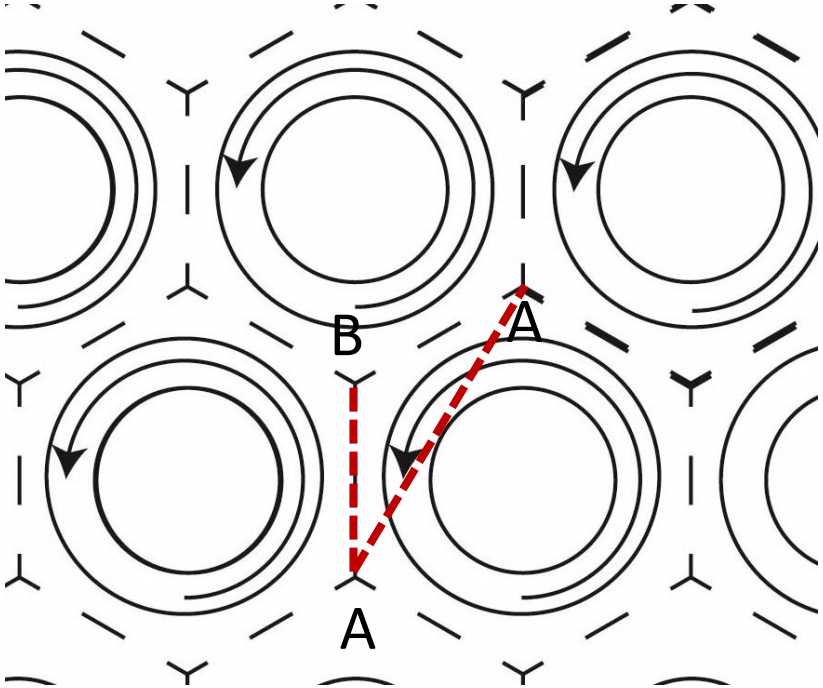
Complex next-nearest-neighbor (NNN) hopping



The first theoretical model of topological insulator beyond quantum Hall effect

Haldane model, PRL 61, 2015-2018 (1988)

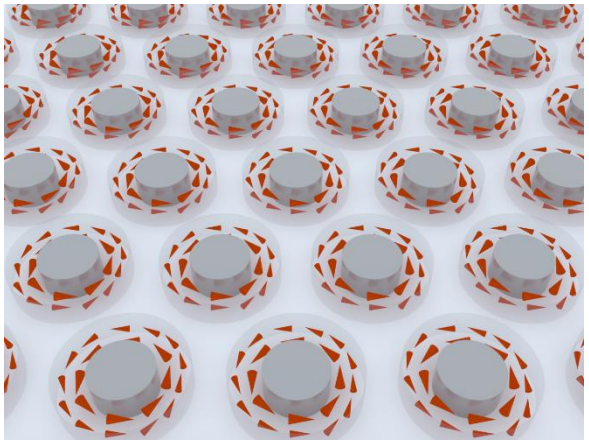
Haldane Model with Staggered Magnetic Field



A-B reciprocal

A-A nonreciprocal

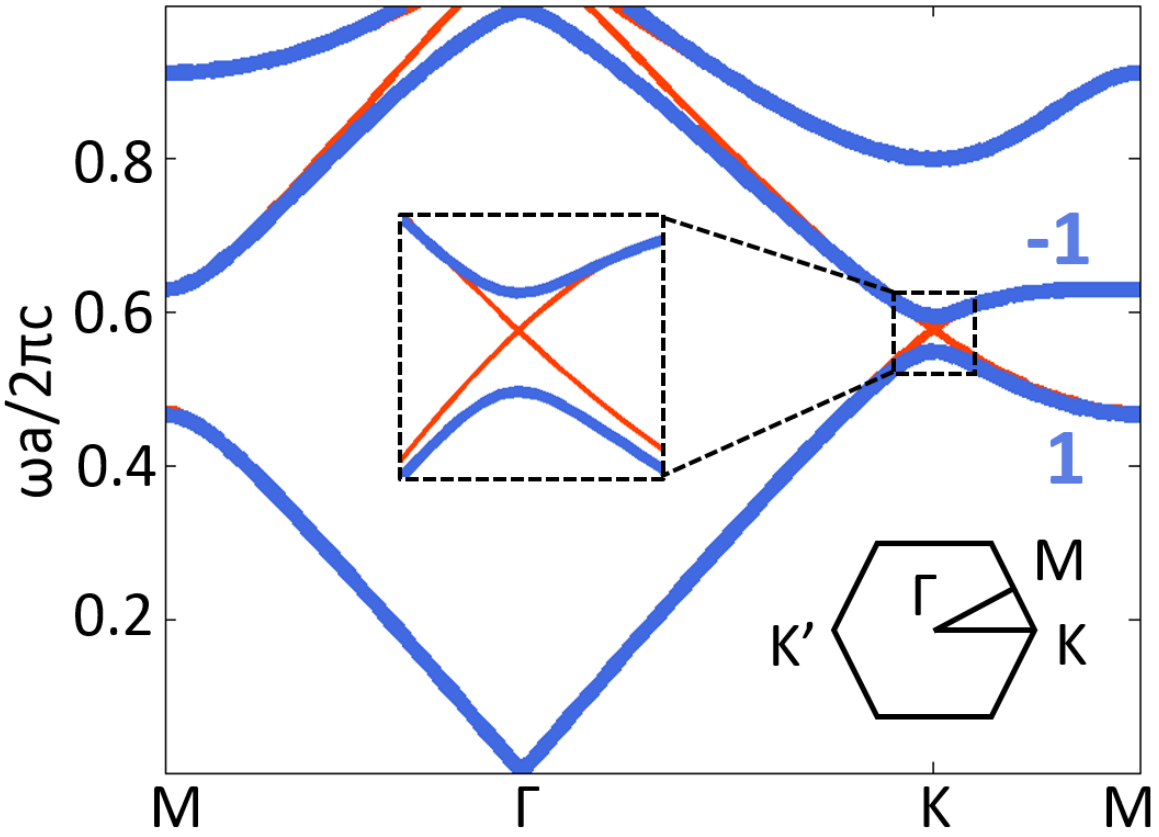
Acoustic Band Structure



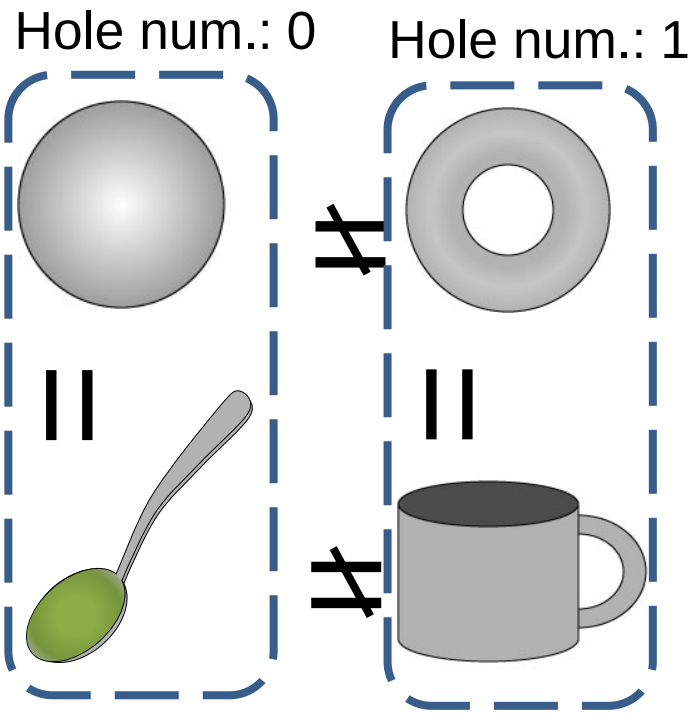
$$\vec{A}_n = i\left\langle \phi_{nk} \mid \nabla_{\vec{k}} \mid \phi_{nk} \right\rangle$$

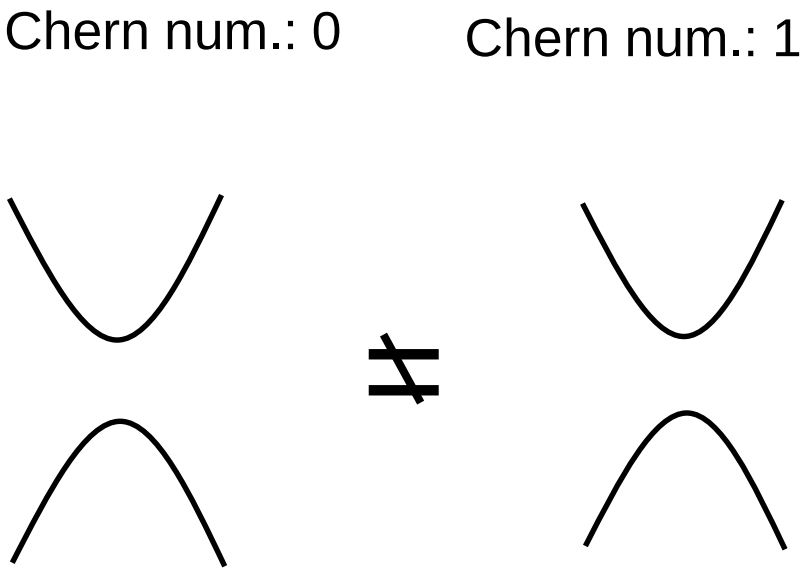
$$C_n = \frac{1}{2\pi}\iint_{BZ} (dk_x dk_y)\nabla_{\vec{k}} \times \vec{A}_n$$

Chern number

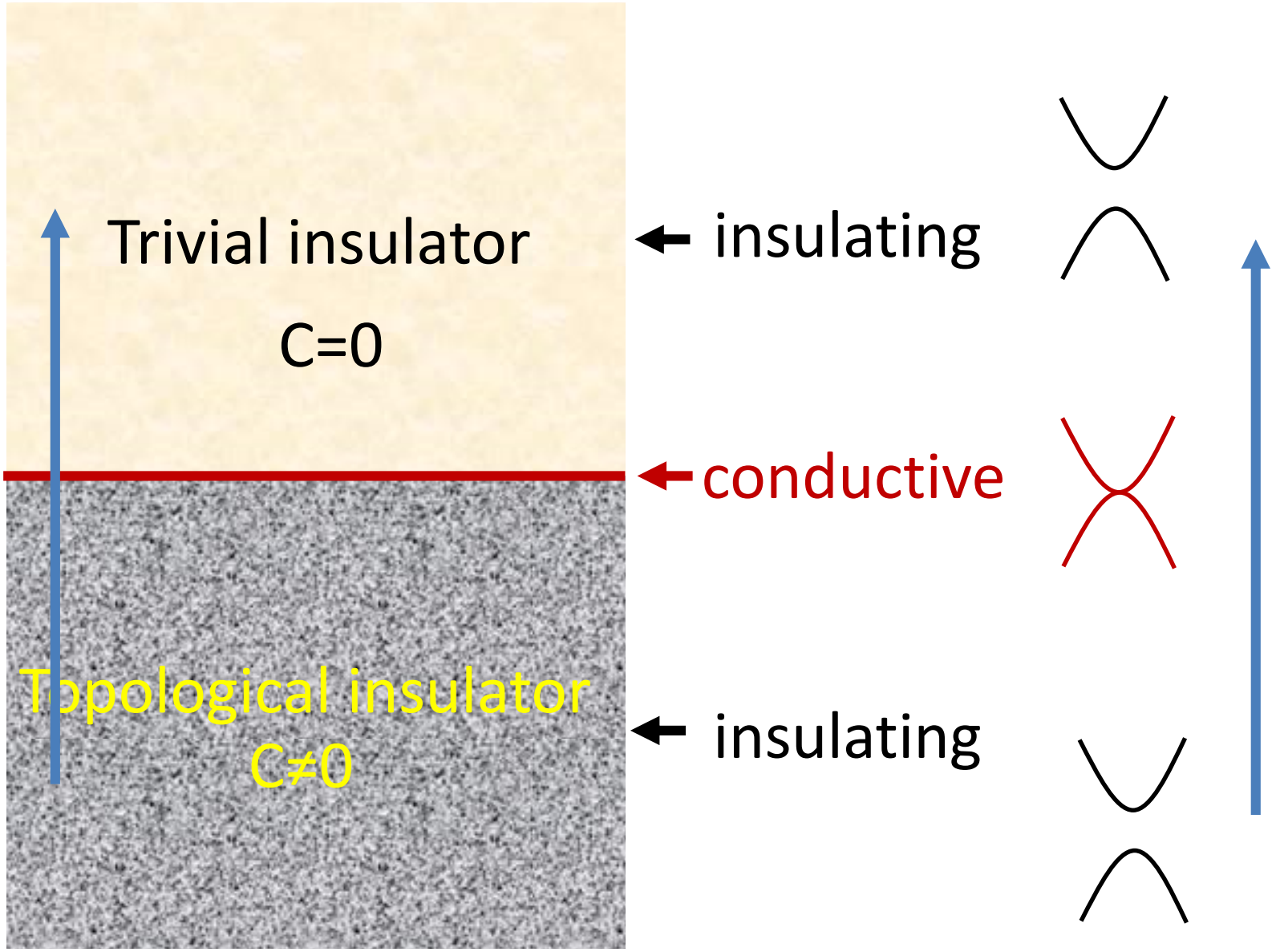


Topology and Topological Insulator



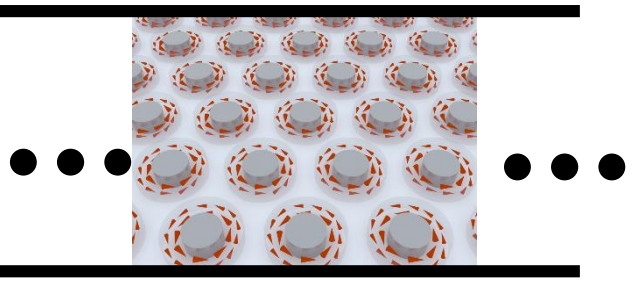


Topology and Topological Insulator



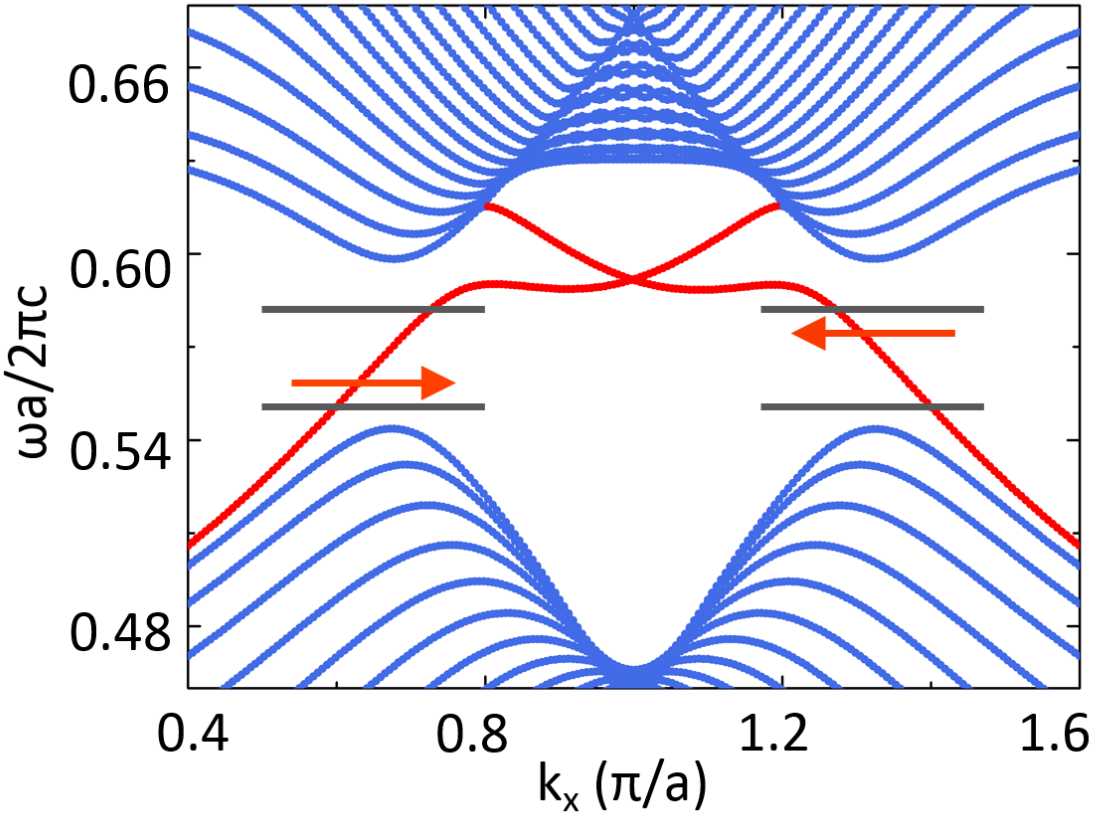
Topological Edge State

A strip



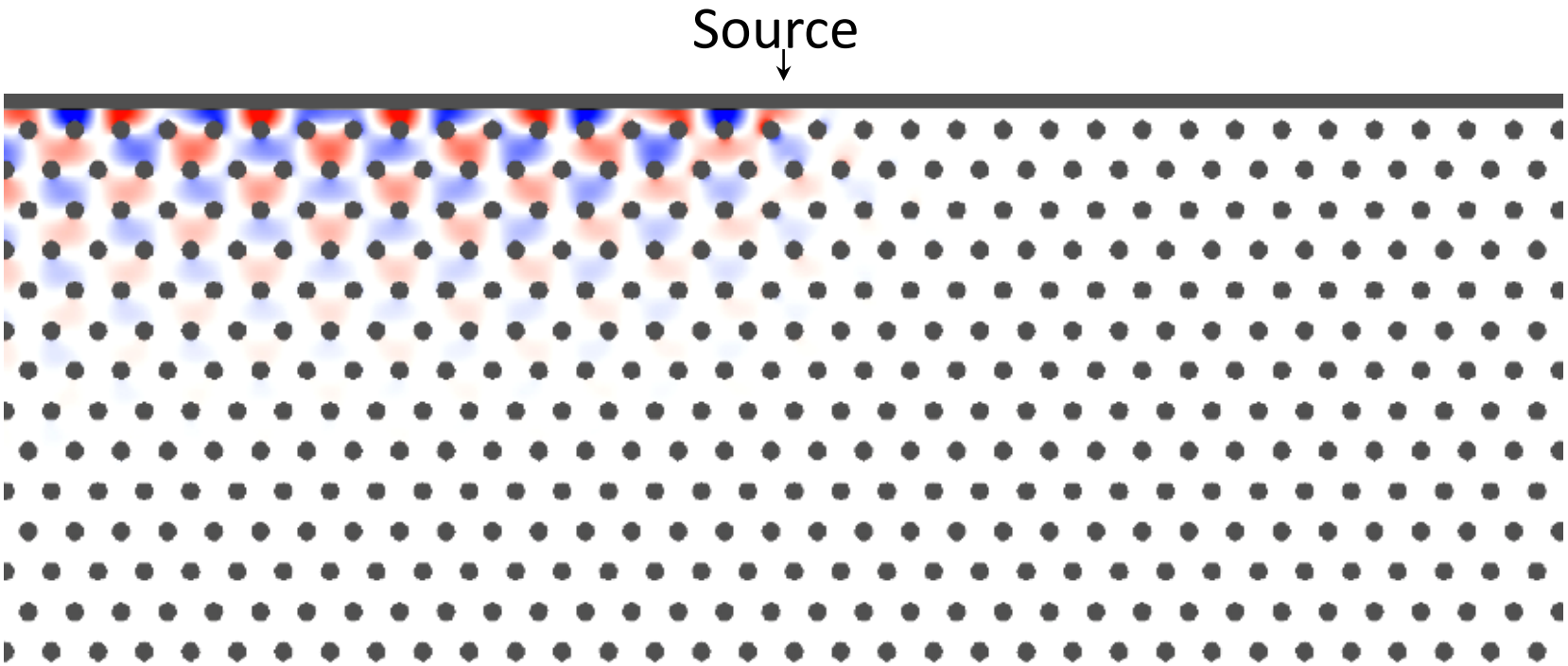
20X1 supercell

$$\Omega = 2\pi \times 400 \text{ rad/s}$$



Topological Edge State

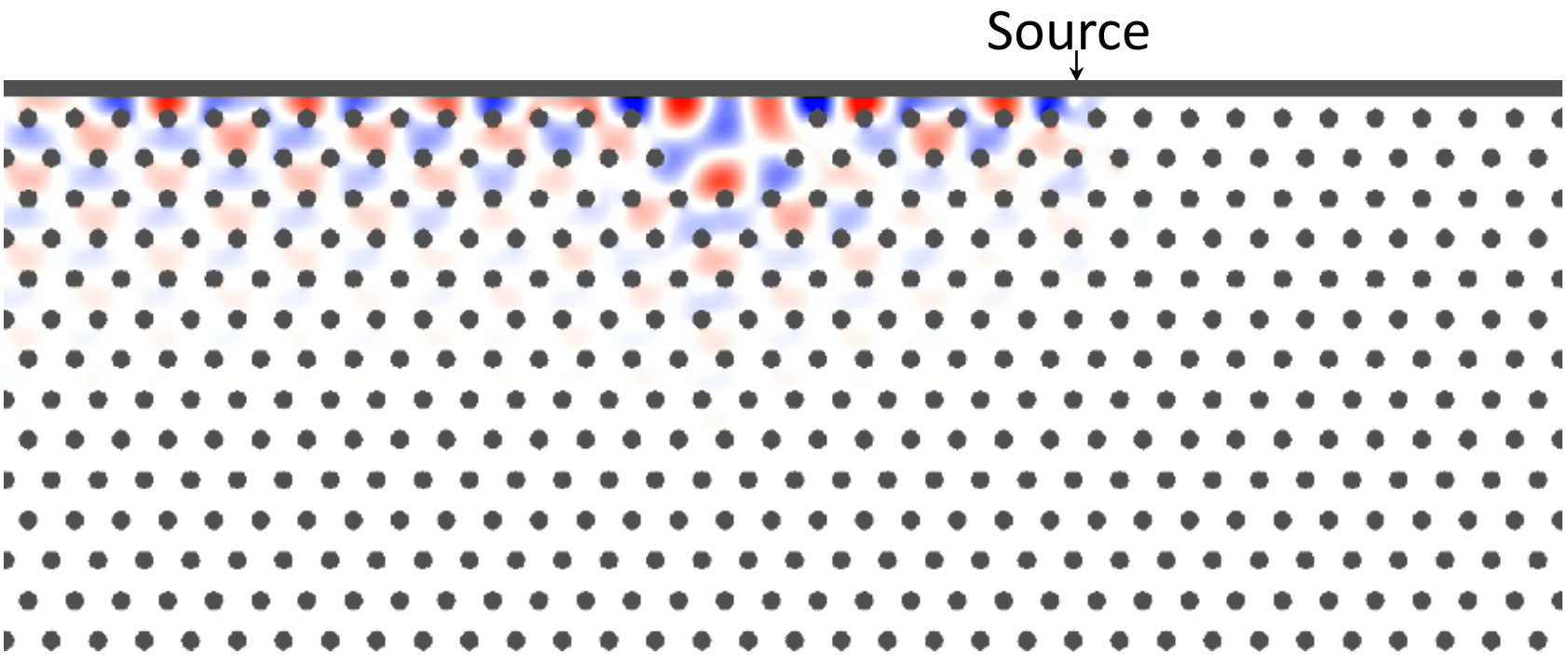
One-way edge state



PRL 114, 114301 (2015)

Topological Edge State

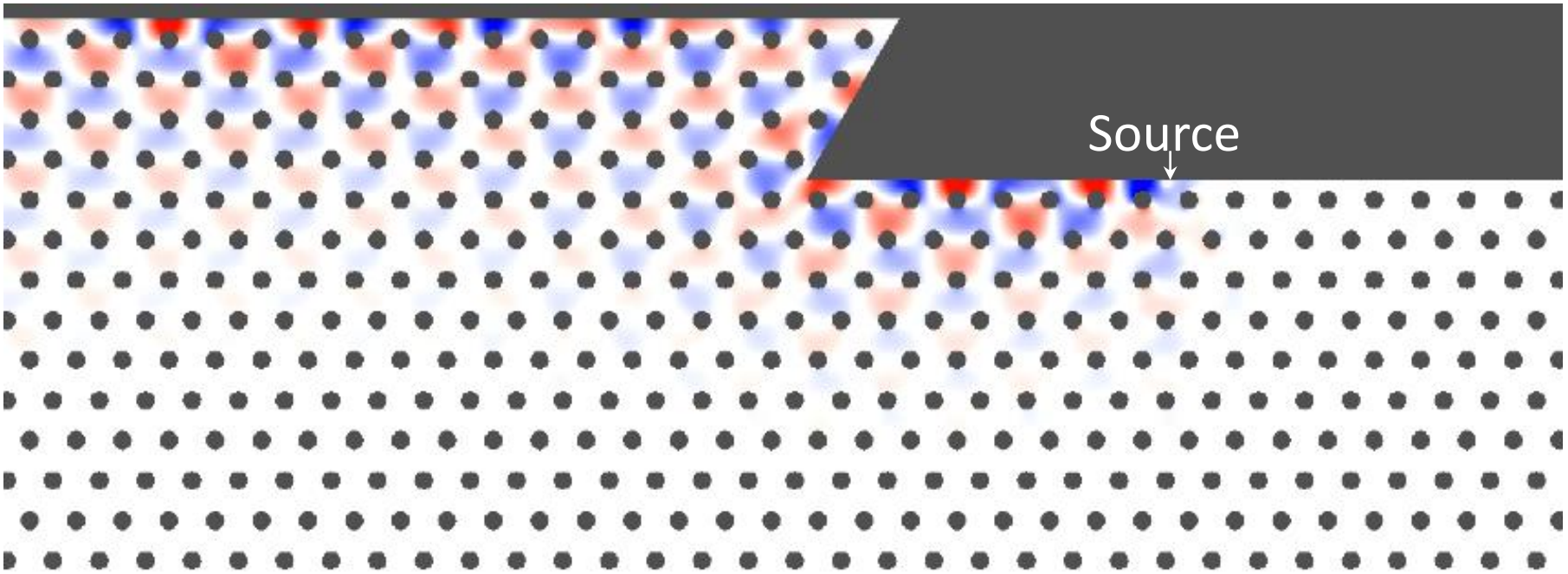
Defect 1: a cavity



PRL 114, 114301 (2015)

Topological Edge State

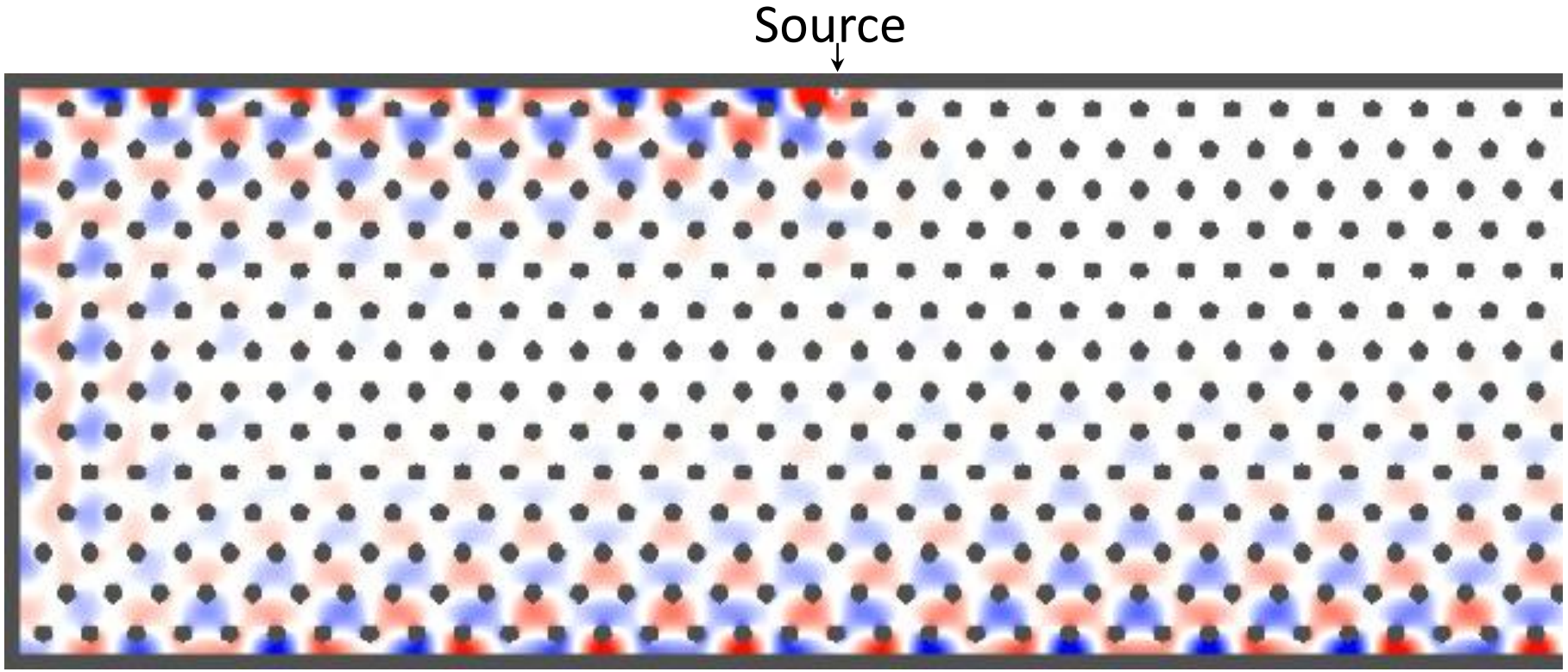
Defect 2: a step



PRL 114, 114301 (2015)

Topological Edge State

Defect 3: sharp corners



PRL 114, 114301 (2015)

Thanks!